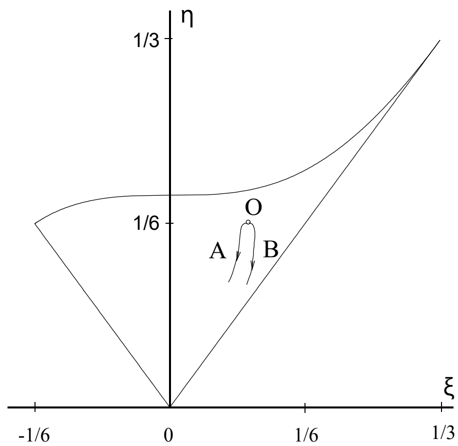

Figure 11.4: Sketch of trajectories (A and B) on the ξ-η plane for two experiments (or DNS) in which the initial spectra are different, but the initial values of **b** are the same. A Reynolds-stress model yields a unique trajectory from initial point O.

Exercise 11.10 Show that Eq. (11.51) implies the following evolution equations for η and ξ:

$$\frac{k}{\varepsilon}\eta\frac{\mathrm{d}\eta}{\mathrm{d}t} = (1 + \tfrac{1}{2}f^{(1)})\eta^2 + \tfrac{1}{2}f^{(2)}\xi^3, \tag{11.62}$$

$$\frac{k}{\varepsilon}\xi^2\frac{\mathrm{d}\xi}{\mathrm{d}t} = (1 + \tfrac{1}{2}f^{(1)})\xi^3 + \tfrac{1}{2}f^{(2)}\eta^4. \tag{11.63}$$

Hence show that F (Eq. 11.35) evolves by

$$\frac{k}{\varepsilon}\frac{\mathrm{d}F}{\mathrm{d}t} = 54[(1 + \tfrac{1}{2}f^{(1)})(3\xi^3 - \eta^2) + \tfrac{1}{2}f^{(2)}(3\eta^4 - \xi^3)]. \tag{11.64}$$

11.4 Rapid Distortion Theory

Homogeneous turbulence can be subjected to time-dependent uniform mean velocity gradients, the magnitude of which can be characterized by

$$\mathcal{S}(t) \equiv (2\bar{S}_{ij}\bar{S}_{ij})^{\frac{1}{2}} \tag{11.65}$$

(see Exercise 5.41 on page 162).[5] As observed above, in turbulent-shear flows, the turbulence-to-mean-shear time scale ratio $\tau\mathcal{S} = \mathcal{S}k/\varepsilon$ is typically

[5]Obviously a different characterization, e.g., $(\bar{\Omega}_{ij}\bar{\Omega}_{ij})^{\frac{1}{2}}$, is needed for solid body rotation in which $\bar{S}_{ij}$ is zero.

in the range 3–6. In contrast, in this Section we consider the *rapid distortion limit* in which $\mathcal{S}k/\varepsilon$ is arbitrarily large. In this limiting case, the evolution of the turbulence is described *exactly* by Rapid Distortion Theory (RDT). RDT provides several useful insights, especially with regard to the rapid pressure–rate-of-strain, models for which are considered in the next section.

11.4.1 Rapid Distortion Equations

In homogeneous turbulence, the fluctuating velocity evolves by (Eq. 5.138)

$$\frac{\bar{\mathrm{D}} u_j}{\bar{\mathrm{D}} t} = -u_i \frac{\partial \langle U_j \rangle}{\partial x_i} - u_i \frac{\partial u_j}{\partial x_i} + \nu \nabla^2 u_j - \frac{1}{\rho}\frac{\partial p'}{\partial x_j}, \tag{11.66}$$

and the Poisson equation for $p' = p^{(r)} + p^{(s)}$ is (Eq. 11.9)

$$\frac{1}{\rho}\nabla^2 (p^{(r)} + p^{(s)}) = -2 \frac{\partial \langle U_i \rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}. \tag{11.67}$$

On the right-hand sides of both of these equations, the first terms represent interactions between the turbulence field $\mathbf{u}$ and the mean velocity gradients; whereas the second terms represent turbulence-turbulence interactions. Given the turbulence field $\mathbf{u}(\mathbf{x}, t)$ at time t, the turbulence-turbulence terms are determined, and are independent of $\partial \langle U_i \rangle / \partial x_j$. On the other hand, the mean-velocity-gradient terms scale linearly with $\mathcal{S}$. Clearly, therefore, in the rapid-distortion limit (i.e., $\mathcal{S} \to \infty$), the terms that scale with $\mathcal{S}$ dominate, all others being negligible in comparison. Hence, in this limit, Eqs. (11.66) and (11.67) reduce to the *rapid distortion equations*

$$\frac{\bar{\mathrm{D}} u_j}{\bar{\mathrm{D}} t} = -u_i \frac{\partial \langle U_j \rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial p^{(r)}}{\partial x_j}, \tag{11.68}$$

and

$$\frac{1}{\rho}\nabla^2 p^{(r)} = -2 \frac{\partial \langle U_i \rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i}. \tag{11.69}$$

The deformation caused by the mean velocity gradients can be considered in terms of the *rate* $\mathcal{S}(t)$, the *amount* (from time 0 to t)

$$s(t) \equiv \int_0^t \mathcal{S}(t')\,\mathrm{d}t', \tag{11.70}$$

and the *geometry* of the deformation

$$\mathcal{G}_{ij}(t) \equiv \frac{1}{\mathcal{S}(t)}\frac{\partial \langle U_i \rangle}{\partial x_j}. \tag{11.71}$$

Note that both s and $\mathcal{G}_{ij}$ are non-dimensional quantities. An interesting feature of rapid distortion theory is that the turbulence field depends on the geometry and the amount of distortion, but it is independent of the rate $\mathcal{S}(t)$—showing that the turbulent viscosity hypothesis is qualitatively incorrect for rapid distortions (Crow 1968). To show this property of the rapid distortion equations, we use s in place of t as an independent variable, and define

$$\tilde{\mathbf{u}}(\mathbf{x}, s) \equiv \mathbf{u}(\mathbf{x}, t), \quad \tilde{\mathcal{G}}_{ij}(s) \equiv \mathcal{G}_{ij}(t), \quad \tilde{p}(\mathbf{x}, s) \equiv \frac{p^{(r)}(\mathbf{x}, t)}{\rho \mathcal{S}(t)}. \tag{11.72}$$

Then (when divided by $\mathcal{S}$) the rapid distortion equations (Eqs. 11.68 and 11.69) become

$$\frac{\bar{\mathrm{D}}\tilde{u}_j}{\bar{\mathrm{D}}s} = -\tilde{u}_i \tilde{\mathcal{G}}_{ji} - \frac{\partial \tilde{p}}{\partial x_j}, \tag{11.73}$$

$$\nabla^2 \tilde{p} = -2\tilde{\mathcal{G}}_{ij} \frac{\partial \tilde{u}_j}{\partial x_i}, \tag{11.74}$$

Given the initial turbulence field $\mathbf{u}(\mathbf{x}, 0)$ and the distortion geometry $\tilde{\mathcal{G}}_{ij}(s)$, these equations can be integrated forward in s to determine the subsequent turbulence field as a function of the amount of distortion s (independent of $\mathcal{S}(t)$ and t). (Having made this observation, we revert to the more familiar variables of Eqs. 11.68 and 11.69.)

To make progress analytically with the rapid distortion equations, it is necessary to circumvent or to solve the Poisson equation for $p^{(r)}$.

In the first works on RDT, Prandtl (1933) and Taylor (1935b) considered the turbulent vorticity equation (the curl of Eq. 11.68), thus eliminating $p^{(r)}$. For irrotational mean distortions ($\bar{\Omega}_{ij} = 0$), this vorticity equation is

$$\frac{\bar{\mathrm{D}}\omega_j}{\bar{\mathrm{D}}t} = \omega_i \frac{\partial \langle U_j \rangle}{\partial x_i} = \omega_i \bar{S}_{ij}. \tag{11.75}$$

Thus, vortex lines (of the *fluctuating* vorticity field) move with, and are stretched by, the *mean* velocity field; and (as in inviscid flow) the vorticity $|\boldsymbol{\omega}|$ increases in proportion to the amount of stretching.

In an axisymmetric contraction with $\bar{S}_{11} > 0$, $\bar{S}_{22} = \bar{S}_{33} = -\frac{1}{2}\bar{S}_{11}$ (see Figs. 10.1 and 10.2 on pages 371 and 372), the vortex lines are tilted towards the x_1 axis, and are stretched in the x_1 direction leading to an intensification of $|\omega_1|$. As a consequence $\langle u_2^2 \rangle$ and $\langle u_3^2 \rangle$ increase relative to $\langle u_1^2 \rangle$, as is observed in Fig. 10.2.